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Classical Behavior After a Phase Transition: II. The Formation of Classical Defects

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Classical defects (monopoles, vortices, etc.) are a characteristic consequence of many phase transitions of quantum fields. Most likely these include transitions in the early universe and such defects would be expected to be present in the universe today. We continue our analysis of the onset of classical behavior after a second-order phase transition in quantum field theory and show how defects appear after such transitions.

KEY WORDS: decoherence; phase transitions; topological defects.

1. INTRODUCTION

Because phase transitions take place in a finite time, causality guarantees that, even for continuous transitions, correlation lengths remain finite. Order parameter fields become frustrated, and topological defects arise so as to reconcile field phases between different correlated regions (Kibble, 1976; Zurek, 1996).

A huge variety of defects is possible, according to the complexity of the initial symmetry and its breaking (Kibble, 1980). Monopoles can easily overwhelm the energy density of the universe, while cosmic strings (vortices) may be the source of high-energy cosmic rays and lensing, as well as contributing to structure formation (although they are not now thought to be its determining factor). More complex strings can exacerbate baryogenesis.

As solutions to the classical field equations, these defects have nonperturbatively large energies, commensurate with the temperature scale at which the transition takes place. Thus, for example, cosmic strings produced in the early universe will be expected to have an energy/unit length (tension) $\sigma \sim (10^{16}\ \text{GeV})^2$, where 10^{16} GeV is the estimated GUT scale. For this reason alone, defects are

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manifestly classical entities after the transition, whose evolution can be handled successfully through the solution of the classical equations that they satisfy (e.g. the Nambu–Goto action for cosmic strings). Nonetheless, their origin in the beginnings of the phase transition that spawned them is quantum mechanical. There are further complications according as the symmetries are global or gauged (Rajantie, 2001), but the simplest of all are topological global vortices (or strings) and it is these that we shall consider here.

Having set up the model, the subsequent sections of this work concern the transition from quantum field theory to classical field theory. We know the mechanism for this, the decohering effects of the quantum environment with which the long-wavelength modes of the order parameter fields interact (Lombardo and Mazzitelli, 1996). Field or phase ordering and classical defect formation are controlled by long-wavelength modes. We consider these modes to constitute the open "system" which undergoes quantum decoherence due to an "environment" which consists of everything else (short-wavelength modes and all other fields interacting with the order parameter fields). This permits the calculation of a decoherence time t_D after the onset of the transition, after which the long-wavelength modes behave classically, subject to generalized Langevin equations. At the same time, the Wigner function can now be interpreted as a Fokker–Planck probability density and we can calculate correlation functions (Lombardo *et al.*, 2000). Given that the correlation functions are now defined in terms of classical probabilities, classical defects can be identified.

The major characteristic of defects is their topological charge. We conclude by showing how localized topological charge precedes the appearance of defects, and gives an estimate of defect densities when they do appear.

For an instantaneous quench the onset of classical behavior has already been discussed by us in an earlier paper in these proceedings, henceforth known as Part I, and elsewhere Lombardo *et al.* (2001), and we shall not repeat all the details here. What we shall do is elucidate those properties of the classicalization due to the environment that are relevant to defect formation.

We conclude by briefly comparing our approach to others for the production of defects. This work has yet to be completed, and will be continued elsewhere.

2. THE MODEL: COUPLING ORDER PARAMETER FIELDS TO THE ENVIRONMENT

The most immediate difference between the model here and that of Part I of this paper lies in the replacement of the single real scalar order parameter field there by a complex field Φ. This is the simplest field permitting acceptable defects, in the form of global vortices. [The defects of a single scalar field with broken $\mathcal{Z}_2$ symmetry, domain walls, would dominate the energy density of the universe if they were present.] As in Part I, we take the Φ field to interact with a collection

of real scalar environmental fields χ_n ($n = 1,\ldots,N$). It is no loss of generality to take all couplings to χ-fields identical, and all χ_n masses identical, to give a model with $U(1) \times O(N)$ or $O(2) \times O(N)$ symmetry broken to $O(N)$. However, the $O(N)$ symmetry of the model, while tactically useful, is accidental and purely for calculational convenience. We adopt a Cartesian field basis $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ because in the linear regime at the early times that are most relevant to us, ϕ_1 and ϕ_2 behave independently. The action is

$$S[\Phi, \chi] = S_{\text{syst}}[\Phi] + S_{\text{env}}[\chi] + S_{\text{int}}[\Phi, \chi], \tag{1}$$

where we have made a provisional separation into system-field and environment with which it interacts, where

$$S_{\text{syst}}[\Phi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + \frac{1}{2}\mu^2 \phi_a^2 - \frac{\lambda}{4}\left(\phi_a^2\right)^2 \right\}$$

is the action for the ϕ_a fields ($a = 1, 2$) (we assume summation over repeated index a), with $\mu^2 > 0$, and

$$S_{\text{env}}[\chi] = \sum_{n=1}^{N} \int d^4x \left\{ \frac{1}{2}\partial_\mu \chi_n \partial^\mu \chi_n - \frac{1}{2}m^2 \chi_n^2 \right\}$$

and

$$S_{\text{int}}[\Phi, \chi] = -\sum_{n=1}^{N} \frac{g}{8\sqrt{N}} \int d^4x \, \phi_a^2(x) \chi_n^2(x),$$

describe the environment-fields χ_n and its interaction with them. These are taken to be weakly coupled, with $1 \gg 1/\sqrt{N} \gg g \simeq \lambda$. Meanwhile, for simplicity the χ_n masses are fixed at $m \simeq \mu$.

The model has a continuous transition for the breaking of the $O(2)$ symmetry at the critical temperature T_c where, in units in which $k_B = 1$, $T_c^2 = \mathcal{O}(\mu^2/\sqrt{N}\lambda) \gg \mu^2$. We shall assume that the initial states of the system and environment are both thermal, at temperatures $T = O(T_c) > T_c$. In this way the Φ field is peaked strongly in field space about the unstable maximum. Incorporating the hard thermal loop "tadpole" diagrams of the χ (and Φ) fields in the Φ mass term leads to the effective action for Φ quasiparticles,

$$S_{\text{syst}}^{\text{eff}}[\Phi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m_\phi^2(T_0)\phi_a^2 - \frac{\lambda}{4}\left(\phi_a^2\right)^2 \right\}$$

where $m_\phi^2(T_0) = -\mu^2(1 - T_0^2/T_c^2) = M^2 > 0$. As in Part I, we can take an initial factorized density matrix at temperature T_0 of the form $\hat{\rho}[T_0] = \hat{\rho}_\Phi[T_0]\hat{\rho}_\chi[T_0]$, where $\hat{\rho}_\Phi[T_0]$ is determined by the quadratic part of $S_{\text{syst}}^{\text{eff}}[\Phi]$ and $\hat{\rho}_\chi[T_0]$ by $S_{\text{env}}[\chi]$. That is, the many χ_n fields have a large effect on Φ, but the Φ-field has negligible

effect on the χ_n. This is crucial for the validity of our approximation. More details are given in Part I.

In fact, with defects in mind, we need to perform a further factorization to isolate the long-wavelength modes of the system-field. As we stressed, the decohering agents are not only the environment-fields χ_n, which have already been introduced but also the short-wavelength modes (Lombardo and Mazzitelli, 1996) of the self-interacting Φ-field.

As in Part I, we split the fields initially as $\Phi = \Phi_< + \Phi_>$ (and $\phi_a = \phi_{<a} + \phi_{>a}$), where the system-field $\Phi_<$ contains the modes with wavelengths longer than the critical value μ^{-1} and the bath or environment-field $\Phi_>$ contains wavelengths shorter than μ^{-1}. This is a natural division, since modes with wavelengths shorter than μ^{-1} are stable and modes with longer wavelengths unstable. Further, the short-wavelengths explore the interior of classical vortices, irrelevant for counting them. However, once the power in the fluctuations has moved to long-wavelength modes, with wavenumber $k_0 < \mu$, the exact delineation is unimportant, and shorter wavelength modes can be included in the "system." With this in mind a more realistic decomposition into 'system' and 'environment' is not (1), but

$$S[\Phi, \chi] = S_{\rm syst}[\Phi_<] + S_{\rm env}[\chi, \Phi_>] + S_{\rm int}[\Phi_<, \chi, \Phi_>],$$

with μ the demarcating momentum, where

$$S_{\rm syst}[\Phi_<] = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi_{<a} \partial^\mu \phi_{<a} + \frac{1}{2}\mu^2 \phi_{<a}^2 - \frac{\lambda}{4}\left(\phi_{<a}^2\right)^2 \right\},$$

and

$$S_{\rm env}[\Phi_>, \chi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi_{>a} \partial^\mu \phi_{>a} - \frac{1}{2}\mu^2 \phi_{>a}^2 - \frac{\lambda}{4}\left(\phi_{>a}^2\right)^2 \right\} + \sum_{n=1}^{N} \int d^4x \left\{ \frac{1}{2}\partial_\mu \chi_n \partial^\mu \chi_n - \frac{1}{2} m^2 \chi_n^2 \right\},$$

$$S_{\rm int}[\Phi_<, \Phi_>, \chi] = -\sum_{n=1}^{N} \frac{g}{8\sqrt{N}} \int d^4x \, \phi_{<a}^2(x) \chi_n^2(x) - \frac{\lambda}{2} \int d^4x \, [\phi_{<a}^2(x)\phi_{>b}^2(x) + 2\phi_{<a}(x)\phi_{<b}(x)\phi_{>a}(x)\phi_{>b}(x)].$$

All terms omitted in the expansion are not relevant for the one-loop calculations for the long-wavelength modes that we shall now consider.

3. THE SIMPLEST TRANSITION: INSTANTANEOUS QUENCH

For simplicity we repeat our assumptions in Part I, in adopting an *instantaneous* temperature quench from T_0 to $T_{\rm f} = 0$ at time $t = 0$, in which $m_\phi^2(T)$ changes

sign and magnitude instantly, concluding with the value $m_\phi^2 = -\mu^2$, $t > 0$ (and beginning with the value $m_\phi^2(T_0) = m^2 = O(\mu^2)$, $t < 0$). We stress that m_ϕ is the renormalized mass, containing the temperature-dependent interactions with all fields.

3.1. The Influence of the Environment

As we observed, the most visible signal that the transition has occurred will be the presence of topological defects, solutions to the *classical* equations of motion $\delta S[\Phi, \chi] = 0$, with $\chi_n = 0$. These are global vortices in the field, around which the field phase θ ($\Phi(\mathbf{x}) = \mathrm{h}(\mathbf{x})\, e^{i\theta(\mathbf{x})}$) changes by 2π. Considered as tubes of 'false' vacuum, with cold thickness $O(\mu^{-1})$, they have energy per unit length $\sigma = O(\mu^2/\lambda)$ (up to multiplicative logarithmic terms $\ln(\mu\xi_{\mathrm{def}})$, where ξ_{def} is vortex separation). In particular, the field Φ vanishes along the vortex core. We can therefore use line-zeroes to track classical defects (Rajantie, 2001; Rivers, 2001). Since classical defects are specific field profiles we need to be able to distinguish between different classical system-field configurations evolving after the transition. As a result, we are only interested in the field-configuration basis for the density matrix.

Since the effects of (bosonic) environments are cumulative each contribution to the environment increases the diffusion term and thereby speeds up the onset of classical behavior. Thus, any part of the environment sets an *upper* bound on t_{D}. For large (but not infinite) N a one-loop approximation is sufficient for calculating t_{D} due to χ fields alone. It is convenient not to have complex arguments and we use ϕ to denote the real Cartesian doublet (ϕ_1, ϕ_2) and χ to denote (χ_1, χ_2, ..., χ_n), etc. At time $t > 0$, the reduced density matrix $\rho_{\mathrm{r}}[\phi_<^+, \phi_<^-, t] = \langle \phi_<^+ | \hat{\rho}_r(t) | \phi_<^- \rangle$ is now

$$\rho_{\mathrm{r}}[\phi_<^+, \phi_<^-, t] = \int \mathcal{D}\chi \int \mathcal{D}\phi_> \, \rho[\phi_<^+, \phi_>, \chi; \phi_<^-, \phi_>, \chi; t], \tag{2}$$

where $\rho[\phi_<^+, \phi_>, \chi; \phi_<^-, \phi_>, \chi; t]$ is the full density matrix, $\mathcal{D}\phi_> = \mathcal{D}\phi_{>1}\mathcal{D}\phi_{>2}$, and $\mathcal{D}\chi = \prod_1^N \mathcal{D}\chi_n$.

The environment will have had the effect of making the system effectively classical once $\rho_{\mathrm{r}}(t)$ is essentially diagonal. This is very different from the late-time dephasing effects found in Habib *et al.* (1996) and Cooper *et al.* (1997), which rely on time-averaged diagonalization. More details are given in Part I and Lombardo *et al.* (2001). Quantum interference can then be ignored and we obtain a classical probability distribution from the diagonal part of $\rho_{\mathrm{r}}(t)$, or equivalently, by means of the reduced Wigner function. For weak coupling (see Part I and Lombardo *et al.*, 2000) there will be no 'recoherence' at later times in which the sense of classical probability will be lost (Antunes *et al.*, 2001).

The temporal evolution of $\rho_r[\phi^+_<, \phi^-_<, t]$ is

$$\rho_r[\phi^+_<, \phi^-_<, t] = \int \mathcal{D}\phi^+_{i<} \int \mathcal{D}\phi^-_{i<} J_r[\phi^+_{f<}, \phi^-_{f<}, t \mid \phi^+_{i<}, \phi^-_{i<}, t_0]\, \rho_r[\phi^+_{i<}, \phi^-_{i<}, t_0],$$

where $\mathcal{D}\phi^+_< = \mathcal{D}\phi^+_{<1}\mathcal{D}\phi^+_{<2}$, etc., and J_r is the reduced evolution operator.

In order to estimate the functional integration that defines the reduced propagator, we perform a saddle point approximation

$$J_r[\phi^+_{f<}, \phi^-_{f<}, t \mid \phi^+_{i<}, \phi^-_{i<}, t_0] \approx \exp i A\big[\phi^{+cl}_<, \phi^{-cl}_<\big],$$

where $\phi^{\pm cl}_<$ is the solution of the equation of motion $\frac{\delta \mathrm{Re} A}{\delta \phi^+_<} \mid_{\phi^+_< = \phi^-_<} = 0$ with boundary conditions $\phi^{\pm cl}_<(t_0) = \phi^\pm_{i<}$ and $\phi^{\pm cl}_<(t) = \phi^\pm_{f<}$. It is very difficult to solve this equation analytically. In order to make it tractable we assume that the system-field contains only one Fourier mode with $\vec{k} = \vec{k}_0$ for the reason indicated earlier, that the long-wavelength modes, for which $|k_0|^2 < \mu^2$, increasingly bunch about a wave number $k_0 < \mu$ which diminishes with time. However, unlike in Part I, we are interested in more than the $k_0 = 0$ mode.

For such small k_0 the classical solution is of the form

$$\phi^{cl}_{<a}(\vec{x}, s) = f_a(s, t)\, \cos(\vec{k}_0 \cdot \vec{x}),$$

where $f_a(s, t)$ satisfies the boundary conditions $f_a(0, t) = \phi_{i<a}$ and $f_a(t, t) = \phi_{f<a}$. Qualitatively, $f_a(s, t)$ grows exponentially with s for $t \le t_{sp}$, and oscillates for $t_{sp} < s < t$ when $t > t_{sp}$. For $t \le t_{sp}$, we approximate it by

$$f_a(s, t) = \phi_{i<a} \frac{\sinh[\omega_0(t - s)]}{\sinh(\omega_0 t)} + \phi_{f<a} \frac{\sinh(\omega_0 s)}{\sinh(\omega_0 t)}, \tag{3}$$

where $\omega_0^2 = \mu^2 - k_0^2$.

We saw in Part I (see also Karra and Rivers, 1997) that the linear approximation is reasonable until almost t_{sp}, where the spinodal time t_{sp} is defined as the time for which $\langle|\Phi_<|^2\rangle_t \sim \eta^2$. That is, t_{sp} is the time it takes for the field to populate the ground states of the model. As a result, t_{sp} is given by

$$\exp[2\mu t_{sp}] \approx \mathcal{O}\left(\frac{\eta^2}{\mu T_c}\right). \tag{4}$$

The exponential factor in Eq. (4), as always, arises from the growth of the unstable long-wavelength modes. The factor T_c^{-1} comes from the $\coth(\beta\omega/2)$ factor that encodes the initial Boltzmann distribution at temperature $T_0 \gtrsim T_c$. Thus,

$$\mu t_{sp} \sim \ln\left(\frac{\eta}{\sqrt{\mu T_c}}\right). \tag{5}$$

As in Part I, it is sufficient to calculate the correction to the usual unitary evolution coming from the noise kernel. For clarity we drop the suffix f on the final

state fields. If $\Delta = (|\Phi^+|^2 - |\Phi^-|^2)/2$ for the *final* field configurations, then the master equation for $\rho_r(\phi_<^+, \phi_<^-, t)$ is

$$i\dot{\rho}_r = \langle \phi_<^+ | [H, \hat{\rho}_r] | \phi_<^- \rangle - iV\Delta^2 D(k_0, t)\rho_r + \cdots \tag{6}$$

The time-dependent diffusion coefficient $D_\chi(k_0, t)$ that determines the effect of the χ fields on the onset of classical behavior acquires a contribution $D_n(k_0, t)$ from each field χ_n,

$$D_n(k_0, t) = \frac{g^2}{16N} \int_0^t ds\, u(s, t) \left[\mathrm{Re}\, G_{++}^2(2k_0; t-s) + 2\, \mathrm{Re}\, G_{++}^2(0; t-s) \right]. \tag{7}$$

where for the case of an instantaneous quench, $u(s, t) = \cosh^2 \omega_0(t - s)$ when $t \le t_{sp}$, and is an oscillatory function of time when $t > t_{sp}$. The G_{++} are the long-wavelength correlation functions of the χ fields for the appropriate contours in the closed time-path.

The contribution from the explicitly environmental χ fields, $D_\chi(k_0, t) = \sum_n D_n(k_0, t)$, takes the form

$$D_\chi(k_0, t) \sim \frac{g^2 T_0^2}{\mu^3} \omega_0 \exp[2\omega_0 t], \tag{8}$$

largely from the end-point behavior at $s = 0$ of the integral (7). For $t > t_{sp}$, the diffusion coefficient stops growing and oscillates around $D(k_0, t = t_{sp})$.

The environment is also constituted by the short-wavelength modes $\Phi_>$ of the self-interacting field. This gives an additional one-loop contribution to $D(k_0, t)$ with the same $u(s)$ but a G_{++} constructed form the short-wavelength modes of the Φ-field as it evolves from the top of the potential hill. Omitting this mode in its calculation just means that any decoherence time t_D obtained from the χ_n alone will be an upper bound on the true decoherence time. An estimate of the Φ-field contribution, based on the inclusion of tadpole diagrams alone, suggests that, with no $1/N$ damping, the $\Phi_>$ modes have the same effect on the dissipation, qualitatively, as all the environmental fields χ_n put together. At an order of magnitude level we can ignore the $\Phi_>$ modes in the calculation of t_D, for which the χ fields alone give a strong bound. We cannot ignore them in the calculation of the defect density, since it is the coarse-graining of the order parameter field that renders the line-zero density finite.

We estimate t_D by solving (in the one-loop approximation) for the off-diagonal elements of the reduced density matrix

$$\rho_r[\phi_<^+, \phi_<^-; t] \lesssim \rho_r^u[\phi_<^+, \phi_<^-; t] \exp\left[-V\Gamma \int_0^t ds\, D_\chi(k_0, s) \right], \tag{9}$$

where ρ_r^u is the solution of the unitary part of the master equation (i.e. without environment). In (9), $\Gamma = O(\mu^4(\bar{\phi}\delta)^2)$, $\bar{\phi} = (|\Phi_<^+| + |\Phi_<^-|)/2\mu$, and $\delta =$

$(|\Phi_<^+| - |\Phi_<^-|)/2\mu$. V is understood as the minimal volume inside which there is no possibility for coherent superpositions of macroscopically distinguishable states for the field. We take this as $\mathcal{O}(\mu^{-3})$ since μ^{-1} is the thickness of an individual vortex. Inside this volume we do not discriminate between field amplitudes which differ by $\mathcal{O}(\mu)$, and therefore, as in Part I, take $\delta^2 \sim \mathcal{O}(1)$. Similarly, we set $\bar{\phi}^2 \sim \mathcal{O}(\alpha/\lambda)$, where $\lambda \leq \alpha \leq 1$ is to be determined self-consistently.

The decoherence of the long-wavelength k_0-mode by the environment occurs when the nondiagonal elements of the reduced density matrix are much smaller than the diagonal ones. In (9) this corresponds to when

$$1 \gtrsim V\Gamma \int_0^{t_D} ds\, D_\chi(k_0, s), \tag{10}$$

since $\rho_r^u[\phi_<^+, \phi_<^-; t]$ is increasingly independent of δ. See Part I for more details. Because the diagonalization of $\rho_r(t)$ occurs in time as an *exponential* of an *exponential*, decoherence occurs extremely quickly, at time

$$\omega_0 t_D \gtrsim \ln\left(\frac{\eta}{T_c\sqrt{\alpha}}\right). \tag{11}$$

For $\omega \approx \mu$, the value of α is determined as $\alpha \simeq \sqrt{\mu/T_c}$ from the condition that at time t_D, $\langle|\phi|^2\rangle_t \sim \alpha\eta^2$. Since $\alpha \ll 1$, in principle, the field has not diffused far from the top of the hill before it is behaving classically.

It follows from Eq. (11) that the upper bound on t_D and, we assume, t_D itself, increases as $k_0 \to \mu$, although we need a better approximation to see how, in detail. However, we stress that as far as counting vortices is concerned, all that matters is how the power in the field fluctuations is distributed. The distance between defects is the relevant wavelength, and not defect size. With $\omega_0 \approx \mu$ for the relevant $k_0^2 = O(\mu/t_D)$ it follows that $1 < \mu t_D \leq \mu t_{sp}$, with

$$\mu t_{sp} - \mu t_D \simeq \frac{1}{4} \ln\left(\frac{T_c}{\mu}\right). \tag{12}$$

While the environment is very effective at decohering adjacent field configurations, it has much less impact on the diagonal matrix elements

$$P_t[\phi_<] = \langle\phi_<|\rho_r(t)|\phi_<\rangle,$$

which give the relative probability that the field takes the value $\phi_<$ at time t, once the theory is classical. This is all that is needed to calculate equal-time correlation functions of $\phi_<$, and thereby (Halperin, 1981; Liu and Mazenko, 1992) to calculate the density of classical defects by itemizing their zero-field cores.

We know that for very early times, the Gaussian approximation for $\rho_r[\phi_<, \phi_<, t]$ is valid, although we should not interpret it as a probability then. We shall now argue that can we use the approximation $P_t[\phi_<] \simeq \rho_r^u[\phi_<, \phi_<, t]$, compatible with (9), at least until time t_D. That is, although the environment is crucial in

making ρ_t off-diagonal, it has much less effect on the diagonal matrix elements, which are the ones we use for calculation.

4. CLASSICAL EQUATIONS

Our analysis of the onset of decoherence shows that it makes little sense to talk about line-zeroes of the field before $t_{\rm D}$ of Eq. (11), since they would be expected to suffer from quantum interference, as well as having a density that is strongly dependent on the scale separating $\phi_<$ from $\phi_>$.

By the time $t_{\rm sp}$ the Gaussian approximation has broken down, and the Goldstone field phase θ will have decoupled from the heavy h-mode. From here onwards it is the massless Goldstone modes whose causal propagation controls field ordering. This is the last ingredient in turning a line-zero into a classical vortex.

This requires both classical probabilities and classical equations. Let us consider probabilities first. The reduced Wigner function is defined as

$$W_t[\phi_<, \pi_<] = \int \mathcal{D}\eta_< \, e^{i\pi_<\eta_<} \langle \phi_< - \eta_< | \rho_{\rm r}(t) | \phi_< + \eta_< \rangle \tag{13}$$

whereby

$$P[\phi_<]_t = \int \mathcal{D}\pi_< \, W_t[\phi_<, \pi_<] = \langle \phi_< | \rho_{\rm r}(t) | \phi_< \rangle \tag{14}$$

is the probability density for field configurations.

For $t \geq t_{\rm D}$ of (11), $W_t[\phi_<, \pi_<]$ is positive, at least for long wavelengths, and can be identified with the Fokker–Planck probability distribution function $P_t^{\rm FP}[\phi_<, \pi_<]$, from which $P[\phi_<]_t$ can be equally identified as the Fokker–Planck probability

$$P^{\rm FP}[\phi_<]_t = \int \mathcal{D}\pi_< \, P_t^{\rm FP}[\phi_<, \pi_<]. \tag{15}$$

Suppose we can calculate $P[\phi_<]_t$ for times $t \geq t_{\rm D}$. This permits us to calculate the equal-time n-point correlation functions

$$G^{(\rm n)}_{<ab\ldots c}(\mathbf{x_1}, \ldots, \mathbf{x_n}, \mathbf{t}) = \int \mathcal{D}\phi_< \, P_t^{\rm FP}[\phi_<] \phi_{<a}(\mathbf{x}_1) \ldots \phi_{<c}(\mathbf{x}_n). \tag{16}$$

As we know (Halperin, 1981; Liu and Mazenko, 1992), equal-time correlators are all we need to calculate densities of zeroes (line-zeroes, etc.). Moreover, the dominance of $P[\phi_<]_t$ by long-wavelength modes permits the adoption of a single decoherence time $t_{\rm D}$ from Eq. (12). However, if all we are going to use is $P[\phi_<]_t$, the diagonal matrix element of ρ_t, there is no real need to construct the Wigner function. We can just do a calculation of $P[\phi_<]_t$ from the start, along the lines of Boyanovsky *et al.* (1994, 1995, 1998).

To see the appearance of individual defects is more difficult, and requires classical stochastic (Langevin) equations. We have yet to complete the analysis. However, as a starting point let us pretend that the only environment is the χ fields. In the usual fashion, one can regard the imaginary part of δA as coming from a single noise source $\xi(x)$, with a Gaussian functional probability distribution given by (Gleiser and Ramos, 1944; Greiner and Muller, 1997; Lombardo and Mazzitelli, 1996)

$$\mathcal{P}[\xi] = N_\xi \exp\left\{-\frac{1}{2}\int d^4x \int d^4y \xi [g^2 N]^{-1} \xi\right\}, \tag{17}$$

where N_ξ is a normalization factor, and $N(x-y) \propto \mathrm{Re}\, G^2_{++}(x-y)$ for the single χ-loop. Indeed, we can write the imaginary part of the influence action as a functional integral over the Gaussian field $\xi(x)$,

$$\int \mathcal{D}\xi\, \mathcal{P}[\xi] \exp\left[\int d^4x - i\,\{\Delta(x)\xi(x)\}\right]$$
$$= \exp\left\{-i\int d^4x \int d^4y\, [\Delta(x) g^2 N(x,y) \Delta(y)]\right\}.$$

Therefore, the coarse-grained effective action (see Part I for definitions) can be rewritten as

$$A[\phi^+, \phi^-] = -\frac{1}{i} \ln \int \mathcal{D}\xi\; P[\xi] \exp\left\{i\, S_{\mathrm{eff}}[\phi^+, \phi^-, \xi]\right\}, \tag{18}$$

where

$$S_{\mathrm{eff}}[\phi^+, \phi^-, \xi] = \mathrm{Re}\, A[\phi^+, \phi^-] - \int d^4x\, [\Delta(x)\xi(x)]. \tag{19}$$

Taking the functional variation

$$\left.\frac{\delta S_{\mathrm{eff}}[\phi^+, \phi^-, \xi]}{\delta \phi_a^+}\right|_{\phi_a^+ = \phi_a^-} = 0 \tag{20}$$

gives the "semiclassical Langevin equation" for the system-field (Gleiser and Ramos, 1944; Greiner and Muller, 1997; Lombardo and Mazzitelli, 1996), (up to factors $O(1)$)

$$\Box\phi_a(x) - \tilde{\mu}^2\phi_a + \tilde{\lambda}\phi_a(x)\phi_b^2(x) + g^2\phi_a(x)\int d^4y\, K(x,y)\phi_b^2(y) = \phi_a(x)\xi(x), \tag{21}$$

where we have assumed only quadratic interactions, and K is the (assumed) common mass retarded loop, arising from the real part of δA. $\tilde{\mu}$ and $\tilde{\lambda}$ are the 'renormalized' constants by virtue of the coupling with the environment.

To estimate the size of $g^2\phi_a(x)\int d^4y\, K(x,y)\phi_b^2(y)$ in comparison with the $\mu^2\phi_a$ term, we note first that

$$K \propto T^2\mu^2 \sim T_c^2\mu^2 \sim \mu^4/\sqrt{N}\lambda \tag{22}$$

for high T. Further, with relevant distance scales no larger than μ^{-1} then, crudely, at time $t_{\rm sp}$,

$$\begin{aligned} &|g^2\int d^4y\, K(x,y)\,\phi_b^2(y)| \\ &\le g^2\int d^4y\,|K(x,y)|\,\phi_b^2(t_{\rm sp}) \\ &\sim \lambda^2\frac{T_c^2}{\mu}t_{\rm sp}\phi_b^2(t_{\rm sp}) \sim \mu^2\frac{\mu t_{\rm sp}}{N^{1/2}} \ll \mu^2, \end{aligned} \tag{23}$$

for large enough N. Since the bound is even smaller at earlier times this suggests that the dissipative term is negligible, in comparison to the mass term, until time $t_{\rm sp}$. Equivalently, for weak coupling the damping due to the thermal environment has a negligible effect on the quasiparticle mass at these early times.

Even though $\langle\phi_a(x)\xi(x)\rangle \neq 0$,

$$\begin{aligned} \langle\phi_a(x)\xi(x)\rangle &\le \sqrt{\langle\phi_a(x)\phi_a(x)\rangle\langle\xi(x)\xi(x)\rangle} \\ &\le \eta\sqrt{\langle\xi(x)\xi(x)\rangle} \end{aligned} \tag{24}$$

(with no summation over a), which, from the behavior of the noise kernel N, can be bounded as

$$\langle\phi_a(x)\xi(x)\rangle_\xi \le \mu^2\phi_a(x).$$

It follows that ϕ_a satisfies the classical equation

$$\Box\phi_a(x) - \tilde{\mu}^2\phi_a(x) + \tilde{\lambda}\phi_a(x)\phi_b^2(x) = 0 \tag{25}$$

to a good approximation for times $t \lesssim t_{\rm sp}$. For such early times the nonlinear term in (25) can be neglected. For times later than $t_{\rm sp}$ neither perturbation theory nor more general non-Gaussian methods are valid. For example, in quantum mechanical models (without environment) *seventh*-order calculations in a self-consistent δ-expansion (that effectively plays the role of a $1/N$ expansion in large-N calculations (Boyanovsky *et al.*, 1994, 1995, 1998) ceases to work once the symmetry has been fully broken (Jones *et al.*, 2001). For such times we need to solve for classical vortices, rather as we would solve for the evolution of vortex tangles in superfluid ^4He.

In terms of the radial and angular fields Eq. (25) becomes

$$\begin{aligned} \Box h + [-\tilde{\mu}^2 + \tilde{\lambda}h^2 - (\partial_\mu\theta\partial^\mu\theta)]h = 0 \\ \partial_\mu(h^2\partial^\mu\theta) = 0. \end{aligned} \tag{26}$$

At early times we have seen that ϕ_1 and ϕ_2 are independent, and there are no Goldstone modes. However, by the time $t \approx t_{\rm sp}$, when $h^2 \sim \eta^2$ and is slowly varying, it follows from the second of the equations that the Goldstone modes have appeared. More generally, it is through the coupling of the Goldstone and Higgs (h) modes that classical defects appear as solutions to (25), in a way that was denied at early times.

We stress that in deriving Eq. (25) we have not had to restrict ourselves to any particular modes in k-space. However, although Eq. (25) looks to be valid at all times, it is only a sensible equation once field configurations have well-defined probabilities associated to them, i.e. *after* decoherence. Here lies a difficulty in that we have seen that $t_{\rm D}$ depends on wavelength, with shorter wavelengths taking longer to become classical. Although the long wavelengths with most power in their fluctuations, that determine the separation of vortices, have become classical by time $t_{\rm sp}$, this is an ensemble statement that cannot be applied easily to individual vortex solutions to (26). Further, Eq. (25) couples $\phi_<$ to $\phi_>$. The effect of including the short-wavelength modes $\phi_>$ in the environment is, most likely, only a qualitative change at a consistent one-loop level, by augmenting K in (21) with a comparable term K_ϕ. There will be a further comparable nondiagonal term $\lambda^2\phi_b \int K_\phi\, \phi_a\phi_b$. We anticipate that the Langevin equations for the stochastic $\phi_<$ will be just as one would guess from a mode decomposition of (26) with short-wavelength modes discarded, and terms with K and N equally ignorable. This will be pursued elsewhere.

5. CLASSICAL DEFECTS

5.1. Line Densities and the Gaussian Approximation

As we observed, the most visible signal that the transition has occurred will be the presence of classical topological vortices (strings), radial solutions to the equations of motion $\delta S[\phi, \chi] = 0$, with $\chi_n = 0$. These are line defects in the field, around which the field phase θ ($\phi = h\, e^{i\theta}$) changes by 2π. We can therefore use line-zeroes to track classical vortices (Gill and Rivers, 1995; Rajantie, 2001; Rivers, 2001). [If we had broken an $O(3)$ symmetry ($a = 1, 2, 3$) the corresponding defects would be global monopoles, giving qualitatively similar conclusions.]

The *total line-zero density* $\bar{z}(\mathbf{x})$ for the long-wavelength mode fields is (Halperin, 1981)

$$\bar{z_i}(\mathbf{x}) = \delta^2[\phi_<(\mathbf{x})]|\epsilon_{ijk}\partial_j\phi_{<1}(\mathbf{x})\partial_k\phi_{<2}(\mathbf{x})|. \tag{27}$$

With line-zeroes as the intersections of planes of zeroes of ϕ_1 and ϕ_2 separately, we have a phase change of $\pm 2\pi$ around the lines of intersection, but for exceptional circumstances.

From Eqs. (14)–(16) the evolution of the two-field correlator ($G^{(2)}_<(\mathbf{x}, 0, t) = G_<(r, t)$) can be written in terms of the unitary (reduced) density matrix $\rho^{\rm u}_{\rm r}$. Thus,

$$G_<(r, t) = \langle \phi_<(\mathbf{x}) \phi^*_<(\mathbf{0}) \rangle_t = \frac{1}{2} \delta_{ab} \langle \phi_{<a}(\mathbf{x}) \phi_{<b}(\mathbf{0}) \rangle_t$$

is well understood. The unstable long-wavelength modes grow exponentially fast. If the power spectrum of the field fluctuations is defined by

$$G_<(r, t) = \int_{k<\mu} \frac{d^3k}{(2\pi)^3} P(k, t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{28}$$

then $k^2 P(k, t)$ rapidly develops a "Bragg peak" at $k^2 = k_0^2 = \mathcal{O}(\mu/t)$.

While the Gaussian approximation (Halperin, 1981; Lui and Mazenko, 1992) is satisfied, the line-zero ensemble density $n_{\rm zero}(t)$ is determined completely by the *short-distance* behavior of $G_<(r, t)$ as

$$n_{\rm zero}(t) = \langle \bar{z}_i(\mathbf{x}) \rangle_t = \frac{-1}{2\pi} \frac{G''_<(0, t)}{G_<(0, t)}. \tag{29}$$

Since $G_<(r, t)$ has short-wavelength modes removed, $G_<(0, t)$ is finite.

Further, for this period when the self-consistent approximation is valid, the field energy $\langle E \rangle_t$ of the system field $\phi_<$ in a box of volume $\mathcal{V}$ becomes

$$\begin{aligned} \langle E \rangle_t &= \mathcal{V}[\langle |\nabla \phi_<|^2 \rangle_t + \lambda(\eta^2 - \langle |\phi_<|^2 \rangle_t)^2] \\ &= \mathcal{V}[2\pi n_{\rm zero}(t) G_<(0, t) + \lambda(\eta^2 - G_<(0, t))^2] \\ &= 2\pi L_{\rm zero}(t) G_<(0, t) + \mathcal{V}[\lambda(\eta^2 - G_<(0, t))^2]. \end{aligned} \tag{30}$$

As before, χ-field fluctuations are absorbed in the definition of μ^2. Equation (30) is obtained by using (29), and $L_{\rm zero}(t) = \mathcal{V} n_{\rm zero}$ is the total length of line-zeroes, on a scale μ^{-1}, in the box of volume $\mathcal{V}$.

We understand Eq. (30) as follows. Suppose it were valid from time $t = 0$, when $G_<(0, 0) = O(\mu^2)$, until time $t = t_{\rm sp}$. At early times most of the system field energy (proportional to $\mathcal{V}$) is in fluctuations not associated with line-zeroes, arising from the field potential. As time passes their energy density decreases as the system field approaches its posttransition value, becoming approximately zero. In part, this is compensated by the term, arising from the field gradients, proportional to the length $L_{\rm zero}$ of line-zeroes, whose energy per unit length increases from $O(\mu^2)$ to $O(\eta^2)$. At time $t_{\rm sp}$, when the fluctuation energy can be ignored, we find (Kavoussanaki *et al.*, 2000)

$$\langle E \rangle_t \sim L_{\rm zero} \sigma, \tag{31}$$

essentially the energy required to produce a vortex tangle of length $L_{\rm zero}$ (up to $O(1)$ factors from the logarithmic tails). Although these line-zeroes have the topological charge and energy of vortices, they are not yet full-fledged defects.

Initially, $G_<(r, t)$ is very dependent on the value of the cutoff. As a result, line-zeroes are extremely fractal, with a separation proportional to the scale at which they are viewed, and are certainly not candidates for defects. Once the 'Bragg peak' at $k = k_0$ is firmly in the interval $k < \mu$, $n_{\rm zero}(t)$ becomes insensitive to a cutoff $O(\mu^{-1})$. This means that line-zeroes are straight at this scale, although they can be approximately random walks at much larger scales. For sufficiently weak couplings of their density this has happened by time $t_{\rm sp}$. The final coupling of radial to angular modes that turns these proto-vortices into vortices incurs no significant energy change, and $n_{\rm zero}$ of (29) is a reliable guide for the initial vortex density.

6. FINAL COMMENTS

The mechanism for vortex production that we have proposed here has two parts. First, the environment renders the order parameter field classical at early times $t_{\rm D} \lesssim t_{\rm sp}$, by or before the transition is complete. Second, classical defects evolve from line-zeroes whose resultant density can already be inferred in the linear regime, but whose specific attributes are a consequence of the nonlinear Langevin equations at the spinodal time $t_{\rm sp}$.

This is very different from traditional explanations, and we conclude with a brief summary of them.

An early explanation, originally due to Kibble (1976), and still of common currency, is that thermal fluctuations in the Ginzburg regime might lead to the production of vortices, again at early times. The reason is the following: once we are below $T_{\rm c}$, the Ginzburg temperature $T_{\rm G} < T_{\rm c}$ signals the temperature above which there is a significant probability for thermal fluctuations between one degenerate ground state and another on the scale of the correlation length at that temperature. That is, the thermal energy in such a fluctuation matches the energy required to pass over the hump of the unstable minimum. This picture presupposes a slow quench, and cannot be accommodated in the instantaneous quench approximation that we have used here. However, our suggestion that defects only appear at, or about, the spinodal time $t_{\rm sp}$ at a density given by the density of line-zeroes is totally at variance with this picture, and thermal activation is not the relevant mechanism.

In fact, this was recognized early by Kibble himself (Kibble, 1980), with his later emphasis on strong causal bounds. It had been noted earlier (Kibble, 1976) that the field must behave independently in initially space-like separated regions. When these domains with different vacua become causally connected we expect defects to link them. This gives late-time predictions that we cannot address with our early-time analysis. A more powerful variant on this theme identifies the time at which defects first appear as the time the adiabatic (long-distance) correlation decreases at the speed of light. This time, and the resultant density, depends on the quench time $\tau_{\rm Q}$. There are also difficulties with this in that we have seen that the separation of line-zeroes are obtained from the short-distance behavior

of $G_<(r, t)$. The argument was not posed for instantaneous quenches of the type discussed above, which correspond to $\tau_Q = O(\mu^{-1})$ for which, if taken literally, it would give defect formation from time $O(\mu^{-1})$, again incorrect. We are in the process of extending the analysis above to slower quenches to make the comparison with the predictions of Kibble (1980) more useful.

Subsequently, two approaches have been adopted. In the first, motivated by condensed matter physics, for which similar causal arguments apply (Zurek, 1996), phenomenological classical stochastic Langevin equations have been proposed (e.g. see Antunes *et al.*, 1999) for the evolution of the fields from early times. These are the counterpart of the phenomenological time-dependent Ginzburg–Landau equations of condensed matter and assume classical probabilities from the beginning. However, unlike the intermediate-time Langevin equations derived previously, these equations do not have multiplicative dissipation and multiplicative noise (Gleiser and Ramos, 1944; Greiner and Muller, 1997; Lombardo and Mazzitelli, 1996). The second approach is intrinsically quantum mechanical, treating the quantum field as a closed system. Although classical correlations are present in the localization of the Wigner function (Guth and Pi, 1991), even for such closed systems, it is difficult to identify defects easily, given that there are no classical probabilities at early times. At best, there is late-time dephasing (Habib *et al.*, 1996). The mechanism proposed in this paper is totally different.

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